

**Objectives:**

- Practice using properties of definite integrals.
- Compare values of definite integrals.
- Use antiderivatives to evaluate definite integrals.

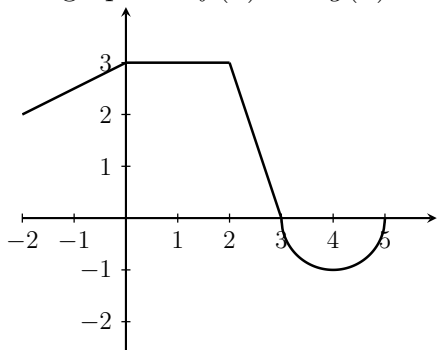
1.  $\int_0^{2\pi} (x + \sin(x)) dx$

$$\begin{aligned} \int_0^{2\pi} x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(0 + i \frac{2\pi}{n}\right) \left(\frac{2\pi}{n}\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4\pi^2}{n^2} i = \lim_{n \rightarrow \infty} \frac{4\pi^2}{n^2} \sum_{i=1}^n i \\ &= \lim_{n \rightarrow \infty} \frac{4\pi^2}{n} \left(\frac{n(n+1)}{2}\right) = \lim_{n \rightarrow \infty} 4\pi \frac{n+1}{n} = \lim_{n \rightarrow \infty} 4\pi \frac{1}{1} = 4\pi \end{aligned}$$

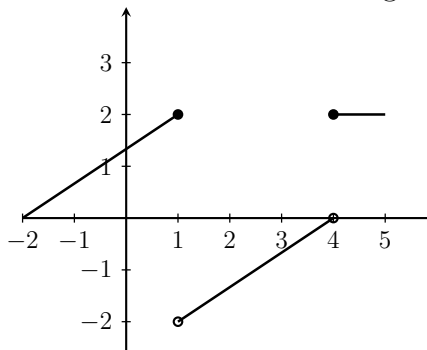
Then draw  $\sin(x)$  on  $[0, 2\pi]$  to see that the area is 0. So,

$$\int_0^{2\pi} (x + \sin(x)) dx = \int_0^{2\pi} x dx + \int_0^{2\pi} \sin(x) dx = 4\pi + 0 = 4\pi$$

2. The graphs of  $f(x)$  and  $g(x)$  are given below. Calculate the integrals.



$f(x)$



$g(x)$

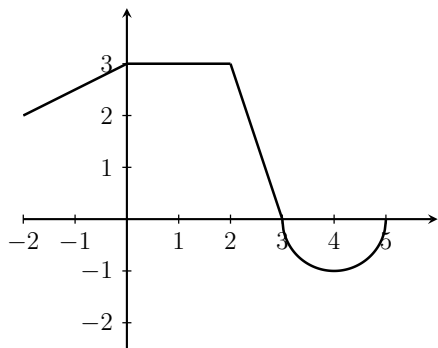
(a)  $\int_{-2}^1 f(x) + g(x) dx = \int_{-2}^1 f(x) dx + \int_{-2}^1 g(x) dx = \left(\frac{2+3}{2} \cdot 2 + 3(1)\right) + \frac{1}{2}(3)(2) = 11$

(b)  $\int_2^5 10f(x) dx = 10 \int_{-2}^1 f(x) dx = 10 \left(\frac{1}{2}(3)(1) + \frac{1}{2}\pi(1)^2\right) = 15 - 5\pi$

(c)  $\int_{-2}^1 g(x) + 5 dx = \int_{-2}^1 g(x) dx + \int_{-2}^1 5 dx = \frac{1}{2}(3)(2) + 5(1 - (-2)) = 18$

(d)  $\int_2^1 g(x) dx + \int_4^2 g(x) dx = \int_4^1 g(x) dx = - \int_1^4 g(x) dx = -\left(\frac{1}{2}(3)(-2)\right) = -(-3) = 3$

**Comparing Integrals:** For the function  $f(x)$  in the previous problem, draw a function  $h(x)$  on the axis such that  $h(x) \geq f(x)$  for all  $x$  values in the interval  $[0, 5]$ :



How does  $\int_0^5 h(x)$  compare to  $\int_0^5 f(x)$ ?

In general we can say that if  $f(x) \leq h(x)$  for all  $x$  in the interval  $[a, b]$ , then  $\int_a^b f(x) \leq \int_a^b h(x)$ .

In particular:

(1) If  $f(x) \geq 0$  for all  $x$  in  $[a, b]$ :

$$\int_a^b f(x) dx \geq 0$$

(2) If  $m \leq f(x) \leq M$  for all  $x$  in  $[a, b]$  where  $m, M$  are constants:

$$m(b - a) \leq \int_a^b f(x) \leq M(b - a)$$

**Example:** It would be very difficult to calculate  $\int_{-2}^3 \sin\left(\frac{1}{x}\right) dx$ . However, we can compare the integral we want to know about to integrals that are easy to compute:

Would probably be useful to draw the function as a reminder of why it's so difficult to integrate.

We know  $-1 \leq \sin(x) \leq 1$ . So,  $\int_{-2}^3 -1 dx \leq \int_{-2}^3 \sin\left(\frac{1}{x}\right) dx \leq \int_{-2}^3 1 dx$ .

From our rectangle knowledge and/or rule from last time, we can see,

$$\begin{aligned} -1(3 - (-2)) &\leq \int_{-2}^3 \sin\left(\frac{1}{x}\right) dx \leq 1(3 - (-2)) \\ -5 &\leq \int_{-2}^3 \sin\left(\frac{1}{x}\right) dx \leq 5 \end{aligned}$$

**Evaluation Theorem (or, Fundamental Theorem of Calculus, Part II)**

If  $f$  is continuous on the closed interval  $[a, b]$

and  $F$  is any antiderivative of  $f$ , (i.e.  $F'(x) = f(x)$ ), then

$$\int_a^b f(x) dx = F(b) - F(a)$$

We use the notation  $F(x) \Big|_a^b$  to denote  $F(b) - F(a)$ .

**Note:**

Hey!! What happens to the  $+C$ ??

The most general antiderivative of  $f(x)$  is  $F(x) + C$ , so why don't we evaluate  $(F(x) + C) \Big|_a^b$  instead? If we do, we get  $(F(b) + C) - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$ . We are not going to leave  $+C$  out of antiderivatives in general, this is a special case.

**Examples**

1.  $\int_{-1}^2 x^4 dx$

$$\int_{-1}^2 x^4 dx = \left. \frac{x^5}{5} \right|_{-1}^2 = \frac{2^5}{5} - \frac{(-1)^5}{5} = \frac{32}{5} + \frac{1}{5} = \frac{33}{5}$$

2.  $\int_0^1 \frac{1}{1+x^2} dx$

$$\int_0^1 \frac{1}{1+x^2} dx = (\arctan(x)) \Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0$$

3.  $\int_2^{10} \left( e^x + 5x - \frac{1}{x} \right) dx$

$$\int_2^{10} \left( e^x + 5x - \frac{1}{x} \right) dx = \left( e^x + \frac{5}{2}x^2 - \ln|x| \right) \Big|_2^{10} = \left( e^{10} + \frac{5}{2}(100) - \ln|100| \right) - \left( e^2 + \frac{5}{2}(4) - \ln|4| \right)$$

Because of this relationship between the integral of  $f(x)$  and the antiderivative of  $f(x)$ , we write  $\int f(x)dx$  to mean the antiderivative of  $f(x)$ . We call this expression an indefinite integral.

**Note:** Key Difference: A definite integral is a number. An indefinite integral is a function. Note that the theorem above says that if  $\int f(x)dx = F(x)$ , then  $\int_a^b f(x)dx = F(b) - F(a)$ .

So now we have 3 ways of calculating an indefinite integral:

(1) a graphical area (2) a limit of Riemann sums and (3) using antiderivatives.

**Interpreting the integral:**

The Evaluation Theorem also appears as the Net Change Theorem.

Since  $F'(x) = f(x)$  is the rate of change of  $F(x)$ , the Evaluation Theorem tell us that

the integral from  $a$  to  $b$  of the rate of change,  $F'(x)$  is equal to  $F(b) - F(a)$ ,

which we call the “net change” of  $F(x)$  from  $a$  to  $b$ .

**Examples:**

If $f(x)$ represents:	Then $\int_a^b f(x)dx = F(b) - F(a)$ represents:
Velocity	Position(b) - Position (a) = “displacement” over $[a, b]$
Marginal Cost	Cost(b) - Cost(a) = net change in cost over $[a, b]$
Growth Rate of a Population	Population(b) - Population(x) = net change in population

**Note:** NET change only measures difference between the value at  $a$  and the value at  $b$ .

E.G., if you leave home at  $t = 0$  hr, drive 10 miles and then get back home at  $t = .5$  hr,  $\int_a^b s(t)dt = 0$ , NOT 10 miles. We’ll talk about how to compute the total distance traveled later.